Mechanical model of normal and anomalous diffusion

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The overdamped dynamics of a charged particle driven by an uniform electric field through a random sequence of scatterers in one dimension is investigated. Analytic expressions of the mean velocity and of the velocity power spectrum are presented. These show that above a threshold value of the field normal diffusion is superimposed to ballistic motion. The diffusion constant can be given explicitly. At the threshold field, the transition between conduction and localization is accompanied by an anomalous diffusion. Our results exemplify that, even in the absence of time-dependent stochastic forces, a purely mechanical model equipped with a quenched disorder can exhibit normal as well as anomalous diffusion, the latter emerging as a critical property. Via another interpretation, as the motion of a particle on an inclined rough surface, our results are relevant for the problem of segregation by flow.

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I. INTRODUCTION

Many physical phenomena like tracers dynamics in flows and granular materials, transport in plasma physics, electronic and heat conduction in disordered media, exhibit anomalous diffusion. These findings have also motivated many theoretical efforts in the direction of understanding more about the possible general mechanism responsible for anomalous diffusion. In most of the cases, this problem has been formulated in terms of stochastic dynamical models as in the celebrated paper by Ya. G. Sinai, describing a random walk in a one-dimensional (1D) random potential [1]. More generally, the dynamics is either given by a master equation, with random hopping rates, describing the effect of a random environment, or, in the continuous version, by a Langevin equation with a time-dependent random force, usually assumed as Gaussian. The literature in this field is so huge that here, we limit ourselves to mention the excellent review article by Bouchaud and Georges [2].

Anomalous as well as normal diffusion have been tackled also in the framework of deterministic dynamics. A well-known example is given by the standard map [3,4], where normal diffusion is observed when the phase space is represented by a chaotic sea with sparse stability island [4,5]. Conversely, when many stability islands coexist finite time trapping of chaotic orbits around them induces correlations yielding anomalous diffusion [5]. Strong anomalous diffusion has been observed in other deterministic systems: here, we just mention two models of 1D intermittent maps introduced by Geisel *et al.* [6] and by Pikovsky [7]. In the former example, the complex scenario of normal and anomalous diffusion emerging from a chaotic dynamics was pointed out. In the latter example, it has been shown that one can pass from normal to anomalous diffusion while varying the polynomial behavior at the unstable fixed point of the intermittent map. It is worth observing that all the above-mentioned examples of deterministic dynamics exhibiting anomalous as well as normal diffusion concern chaotic maps, i.e., time discrete unpredictable dynamics. Another example of diffusion created by a deterministic dynamics, that of a one-dimensional Brownian particle subject to elastic collisions with "light" particles was given in Ref. [8]. Depending on the asymptotic scaling of the mass ratio between the test particle and the other particles, the asymptotic process was shown to be either Ornstein-Uhlenbeck or Wiener.

In this paper, we consider the purely mechanical problem of an overdamped charged particle moving along a line and submitted to an electric field and to a random potential created by a set of quenched random scatterers. At variance with similar models that have been recently investigated [9,10], we do not include any random time-dependent force of the Langevin type.

It is worth stressing that this simple mechanical model can be exactly solved, thus, providing a completely analytic approach to the investigation of the relation between normal and anomalous diffusion, the latter emerging as a sort of "critical" behavior of the former. In fact, we show that if the electric field is larger than a critical value, a current is created and the particle exhibits, on top of the ballistic motion, a diffusive behavior. The diffusion constant can be explicitly computed. It is a function of the average distance between scatterers and of the mean value and the variance of the passage time through a scatterer, and it vanishes, as expected, with vanishing randomness. Below the critical value of the electric field also the current vanishes. The transition from an insulator to a conductor induced by the electric field can be described by analogy with phase transitions as being first or second order. In the former case, the current and the diffusion constant do not vanish and remain finite at the threshold, whereas in the latter case the current vanishes and the diffusion constant diverges according to a power-law behavior. The critical properties can be characterized by a scaling law for the power spectrum of the current. Our study shows that quenched disorder is sufficient to create normal diffusion and also anomalous diffusion, at least for the critical value of the electric field.

In the following section, we derive the general formulas which will be used in Sec. III for solving two particular models. In the first model, that we call a gas, the scatterers are distributed uniformly in an interval. In the second model, which can be considered as a special case of the first one, and called a crystal, they follow each other at equal distances. In Sec. IV, we discuss the transition between conduction and localization through a specific example. The derivation of a scaling function for the velocity power spectrum and some concluding remarks are contained in Sec. V.

There is another problem which can be effectively described by our model. Consider a massive particle submitted to strong friction, sitting or sliding on a rough surface, inclined with a certain angle α with respect to the ground. If we model the roughness of the surface by a lattice of random harmonic traps, we have a two-dimensional version of our model, the component of the gravity force parallel to the surface playing the role of the driving field. In the Appendix, we show how the long-time dynamics is described by the one-dimensional models discussed in this paper. It is a consequence of our results that when α passes a certain critical value the particle is able to fall down indefinitely and then its motion is diffusive. For harmonic traps, the average velocity and the diffusion constant take a nonzero and finite value at the critical angle. Other potential traps can lead to anomalous diffusion at the critical value of the angle. It is worth noting that this type of problem is important in understanding the phenomenon of segregation by flow. A stochastic model has been recently proposed and investigated numerically [11], in order to account for some interesting experimental studies on this topic [12]. Our model can provide a theoretical interpretation of this phenomenon based on an analytic approach.

II. GENERAL SETUP

The equation of motion of an overdamped particle in a constant external field $E>0$ is

$$\dot{x}=E-V'(x). \tag{1}$$

Throughout the paper, we assume the potential to be a superposition of disjoint scatterers,

$$V(x)=\sum_{j=1}^{N}\phi_j(x-r_j), \quad \phi_j(x)=0 \quad \text{if} \quad |x|>a, \tag{2}$$

with centers r_j in an interval $[L_0,L_1]$, subject to the constraints

$$r_{j+1}-r_j\geqslant\sigma\geqslant 2a. \tag{3}$$

This also implies $L_1-L_0\geqslant(N-1)\sigma$. We introduce the definitions

$$r_j^{\pm}=r_j\pm a \tag{4}$$

and

$$t_j^{\pm}: \quad x(t_j^{\pm})=r_j^{\pm}, \tag{5}$$

and assume

$$t_j^-<t_j^+<t_{j+1}^-, \tag{6}$$

that is, the particle moves steadily from left to right, "above" the barriers. This imposes a condition on the external field, namely,

$$E>\phi_j'(x) \quad \text{for all } j \text{ and } x. \tag{7}$$

The particle starts at $t=0$ in $x(0)=r_0<r_1^-$ and for $t<t_1^-$ it moves with a constant speed $\dot{x}=E$, so that

$$x(t)=r_0+Et. \tag{8}$$

By solving for t, we obtain

$$t_1^-=\frac{r_1^--r_0}{E}. \tag{9}$$

If $t_j^-<t<t_j^+$, i.e., $r_j^-<x<r_j^+$, the equation of motion reads

$$\frac{d}{dt}(x-r_j)=E-\phi_j'(x-r_j), \tag{10}$$

whose solution is

$$t-t_j^-=\int_{-a}^{x-r_j}\frac{d\eta}{E-\phi_j'(\eta)}=:\vartheta_j(x-r_j). \tag{11}$$

In particular, the passage time above the jth scatterer is

$$\tau_j:=t_j^+-t_j^-=\vartheta_j(a)=\int_{-a}^{a}\frac{d\eta}{E-\phi_j'(\eta)}. \tag{12}$$

The solution between successive bumps $r_j^+<x<r_{j+1}^-$, i.e., $t_j^+<t<t_{j+1}^-$ again corresponds to the free case $\dot{x}=E$, so that

$$t_{j+1}^--t_j^+=\frac{r_{j+1}^--r_j^+}{E}. \tag{13}$$

Finally, for $t>t_N^+$ we are again in the free case and integration yields

$$x=r_N^++E(t-t_N^+). \tag{14}$$

For the running solutions that we are looking for, the overall equation of motion can be put in the form

$$\dot{x}=E-\sum \phi_j'[x(t)-r_j]\chi_{(t_j^-,t_j^+)}(t), \tag{15}$$

where χ denotes the characteristic function. The Laplace transform of this equation reads

$$\int_0^\infty \dot{x}(t)e^{-\mu t}dt=\frac{E}{\mu}-\sum_{j=1}^{N}\int_{t_j^-}^{t_j^+}e^{-\mu t}\phi_j'[x(t)-r_j]dt. \tag{16}$$

Substituting $\phi_j'(x-r_j)$ from Eq. (10) and using the definitions (11), (12), and

$$\alpha_j(\mu)=\int_{-a}^{a} e^{-\mu\vartheta_j(y)}dy, \tag{17}$$

we obtain

$$\int_0^\infty \dot{x}(t)e^{-\mu t}dt=\frac{E}{\mu}+\sum_{j=1}^{N} e^{-\mu t_j^-}\left[\alpha_j-\frac{E}{\mu}(1-e^{-\mu\tau_j})\right], \tag{18}$$

or, with

$$C_j(\mu)=\mu\alpha_j(\mu)-E(1-e^{-\mu\tau_j}), \tag{19}$$

$$\mu\int_0^\infty \dot{x}(t)e^{-\mu t}dt=E+\sum_{j=1}^{N} e^{-\mu t_j^-}C_j(\mu). \tag{20}$$

Time average of the velocity will be obtained by sending first N to infinity and then μ to zero in the last equation. Since the distance freely run over by the particle up to time t_j^- is $r_j-r_0-a-(j-1)2a$, we obtain

$$t_j^-=\sum_{k=1}^{j-1}\tau_k+\frac{r_j}{E}-\frac{(2j-1)a+r_0}{E}. \tag{21}$$

We can take r_j random or not: if they are random the condition (3) must be fulfilled. Also ϕ_j can be random or not: if they are random we suppose they are identically distributed and independent, and also independent of all r_i. By averaging Eq. (20) over disorder and noticing that $e^{-\mu t_j^-}$ is independent of $C_j(\mu)$, and $C_j(\mu)$ are identically distributed, we obtain

$$\mu\int_0^\infty \overline{\dot{x}(t)e^{-\mu t}}dt=E+\overline{C(\mu)}\sum_{j=1}^{N}\overline{e^{-\mu t_j^-}}\equiv E+B_N(\mu). \tag{22}$$

In what follows, we drop the subscript of averaged quantities whenever the average is independent of the subscript. By using Eq. (21), we can write

$$\overline{e^{-\mu t_j^-}}=e^{(\mu/E)[(2j-1)a+r_0]}D(\mu)^{j-1}\overline{e^{-\mu r_j/E}},\quad D(\mu)=\overline{e^{-\mu\tau}}. \tag{23}$$

The average factorizes and becomes a power because τ_k and hence, $e^{-\mu\tau_k}$ are independent and identically distributed.

A relevant quantity we are looking for is the time-autocorrelation function of the velocity. The average of the Laplace transform of this quantity can be inferred from Eqs. (20) and (22) and reads

$$\begin{aligned}\mu\mu'\int_0^\infty\int_0^\infty & e^{-\mu t-\mu' t'}[\overline{\dot{x}(t)\dot{x}(t')}-\overline{\dot{x}(t)}\,\overline{\dot{x}(t')}]dtdt'\\ &=\sum_{j,j'=1}^{N}[\overline{e^{-\mu t_j^-}e^{-\mu' t_{j'}^-}C_j(\mu)C_{j'}(\mu')}\\ &\quad-\overline{e^{-\mu t_j^-}}\,\overline{C(\mu)}\,\overline{e^{-\mu' t_{j'}^-}}\,\overline{C(\mu')}\\ &=\overline{C(\mu)C(\mu')}\sum_{j=1}^{N}\overline{e^{-(\mu+\mu')t_j^-}}\\ &\quad+\Lambda(\mu,\mu')+\Lambda(\mu',\mu)-B_N(\mu)B_N(\mu').\end{aligned} \tag{24}$$

Here,

$$\begin{aligned}\Lambda(\mu,\mu')&=\sum_{1\le j<j'\le N}\overline{e^{-\mu t_j^- -\mu' t_{j'}^-}C_j(\mu)C_{j'}(\mu')}\\ &=\overline{C(\mu')}\,\overline{e^{-\mu'\tau}C(\mu)}\\ &\quad\times\sum_{1\le j<j'\le N}e^{(\mu/E)[(2j-1)a+r_0]}\\ &\quad\times e^{(\mu'/E)[(2j'-1)a+r_0]}\overline{e^{-(\mu r_j+\mu' r_{j'})/E}}\\ &\quad\times D(\mu+\mu')^{j-1}D(\mu')^{j'-j-1}.\end{aligned} \tag{25}$$

III. MODELS

A. A gas of scatterers

In the first model, we are going to consider the scatterers are uniformly distributed in the interval $[L_0,L_1]$, but respect the constraints (3). We introduce a set of random variables $\{x_j\}$ through

$$r_j=L_0+(j-1)\sigma+x_j, \tag{26}$$

The joint probability density of $x_1,\ldots,x_N$ is chosen to be

$$p(x_1,\ldots,x_N)=\frac{N!}{L^N}\prod_{j=0}^{N}\theta(x_{j+1}-x_j), \tag{27}$$

where $x_0=0$ and $x_{N+1}=L:=L_1-L_0-(N-1)\sigma$. Then p is two valued and nonzero if and only if $0\le x_1\le x_2\le\cdots\le x_N\le L$.

To compute the average over $x_1,\ldots,x_N$ it is useful to introduce

$$\rho_j(x):=\overline{\delta(x-x_j)}=\frac{N!}{L^N}\frac{x^{j-1}}{(j-1)!}\frac{(L-x)^{N-j}}{(N-j)!}, \tag{28}$$

and for $j<j'$

$$\begin{aligned}\rho_{jj'}(x,x')&:=\overline{\delta(x-x_j)\,\delta(x'-x_{j'})}\\ &=\frac{N!}{L^N}\frac{x^{j-1}}{(j-1)!}\frac{(x'-x)^{j'-j-1}}{(j'-j-1)!}\frac{(L-x')^{N-j'}}{(N-j')!},\end{aligned} \tag{29}$$

First, we calculate the mean asymptotic velocity. From Eqs. (23) and (26), we obtain

$$\sum_{j=1}^{N}\overline{e^{-\mu t_j^-}}=e^{-\mu c/E}\int_0^L dx\,e^{-\mu x/E}\sum_j\rho_j(x)e^{-\beta(\mu)(j-1)}, \tag{30}$$

where $c=L_0-a-r_0$ and

$$e^{-\beta(\mu)} \equiv e^{-(\mu/E)(\sigma-2a)} D(\mu). \tag{31}$$

We can choose $r_0 = L_0 - a$, and thus $c=0$, without restricting generality. Summation over j can be performed by the use of the binomial formula. It yields

$$\sum_{j=1}^{N} \overline{e^{-\mu t_j^-}} = \frac{N}{L} \int_0^L dx\, e^{-\mu/Ex} \left[1 - \frac{x}{L}(1-e^{-\beta})\right]^{N-1}. \tag{32}$$

We send N and $L_1 - L_0$ to infinity so that the mean distance ℓ exists and $\ell = \lim(L_1 - L_0)/N > \sigma$. This yields $\lim N/L = (\ell-\sigma)^{-1}$ and

$$\sum_{j=1}^{\infty} \overline{e^{-\mu t_j^-}} = \left[\frac{\mu}{E}(\ell-\sigma) + 1 - e^{-\beta(\mu)}\right]^{-1}. \tag{33}$$

With Eq. (22) and the notation

$$B(\mu) = \lim B_N(\mu) = \overline{C(\mu)} \left[\frac{\mu}{E}(\ell-\sigma) + 1 - e^{-\beta(\mu)}\right]^{-1}, \tag{34}$$

we can write

$$\mu \int_0^\infty \overline{\dot{x}(t) e^{-\mu t}} = E + B(\mu). \tag{35}$$

When μ goes to zero, we asymptotically find

$$\begin{aligned} \overline{C(\mu)} &= \mu[2a - E\overline{\tau}], \\ \beta(\mu) &= \mu[\overline{\tau} + (\sigma-2a)/E], \end{aligned} \tag{36}$$

and the limit of Eq. (35) is, therefore,

$$\overline{\dot{x}(\infty)} = \frac{E\ell}{\ell + E\overline{\tau} - 2a}. \tag{37}$$

This is the time average of the velocity over an infinite run, that is,

$$\overline{\dot{x}(\infty)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \dot{x}(t)\,dt = \lim_{T\to\infty} \frac{x(T)-x(0)}{T}. \tag{38}$$

Remarkably, this is different from the average of $\dot{x}(x)$ over an infinite distance,

$$\overline{v(x)} \equiv \lim_{r\to\infty} \frac{1}{r} \int_{r_0}^{r_0+r} [E - V'(x)]\,dx = E, \tag{39}$$

which is the same as in the absence of the random potential. The reason is that the work of each scatterer exerted on the particle is vanishing,

$$\phi_j(r_j^+ + 0) - \phi_j(r_j^- - 0) = 0.$$

Notice that the result (37) could have been obtained without any computation: $\overline{\dot{x}(\infty)}$ is the average velocity over any interval of length ℓ containing (the support of) a single scatterer.

Before going further, let us make a comment on the order of averaging, we have chosen to compute the mean velocity. The integral on the left-hand side of Eq. (20) corresponds to a Cesaro time average of $\dot{x}(t)$ over a finite-time interval $1/\mu$ for a given realization of the potential. To perform *first* the finite-time average was essential to obtain the representation in the form of a sum on the right-hand side of the same equation. Then, we could take the ensemble average and, finally, with μ going to zero, extend time average to infinite time. In Eq. (20), we could stop at a finite N because every term depended only on the randomness of the preceding terms. In Eq. (20), we could equally have extended the summation to the full sequence. Indeed, under some weak hypothesis on the distribution of ϕ_j the sequence τ_j is separated from zero for every realization, and therefore, there exists some $c>0$ such that $t_j^- \geqslant cj$. Since $C_j(\mu) \leqslant 2a\mu$, the infinite sum is absolutely convergent and, thus, ensemble average commutes with infinite summation. It may also be that this latter is self-averaging, but we do not discuss this question here.

To compute the time-autocorrelation function of the velocity, we need to evaluate $\Lambda(\mu,\mu')$. We first rewrite it as

$$\begin{aligned} \Lambda(\mu,\mu') = \overline{C(\mu')} \overline{e^{-\mu'\tau} C(\mu)} & \\ \times \sum_{1 \leqslant j < j' \leqslant N} & e^{-(\mu/E)(\sigma-2a)(j-1)} \\ \times & e^{-(\mu'/E)(\sigma-2a)(j'-1)} \int_0^L dx \int_x^L dx' \\ \times & \rho_{jj'}(x,x') e^{-\mu/Ex} e^{-\mu'/Ex'} \\ \times & D(\mu+\mu')^{j-1} D(\mu')^{j'-j-1}. \end{aligned} \tag{40}$$

With Eq. (31),

$$\begin{aligned} \Lambda(\mu,\mu') = \overline{C(\mu')} \overline{e^{-\mu'\tau} C(\mu)} e^{-(\mu'/E)(\sigma-2a)} \int_0^L dx & \\ \times \int_x^L dx'\, e^{-\mu/Ex} e^{-\mu'/Ex'} \sum_{1 \leqslant j < j' \leqslant N} & \rho_{jj'}(x,x') \\ \times e^{-\beta(\mu+\mu')(j-1)} e^{-\beta(\mu')(j'-j-1)}. & \end{aligned} \tag{41}$$

Inserting $\rho_{jj'}$ from Eq. (29) and using the trinomial formula

$$\sum_{0 \leqslant l \leqslant m \leqslant M} \frac{M!}{l!(m-l)!(M-m)!} a^l b^{m-l} c^{M-m} = (a+b+c)^M, \tag{42}$$

with $l=j-1$, $m=j'-2$, $M=N-2$,

$$a = e^{-\beta(\mu+\mu')} x, \quad b = e^{-\beta(\mu')}(x'-x), \quad c = L - x',$$

we get

$$\Lambda(\mu,\mu')=\frac{N(N-1)}{L^2}\overline{C(\mu')\overline{e^{-\mu'\tau}C(\mu)}e^{-(\mu'/E)(\sigma-2a)}}$$
$$\times\int_0^L dx\int_x^L dx' e^{-\mu/Ex}e^{-\mu'/Ex'}\left\{1+\frac{1}{L}\right.$$
$$\left.\times[e^{-\beta(\mu+\mu')}x+e^{-\beta(\mu')}(x'-x)-x']\right\}^{N-2}. \tag{43}$$

In the limit of N and L going to infinity this yields

$$\Lambda(\mu,\mu')=(\ell-\sigma)^{-2}\overline{C(\mu')\overline{e^{-\mu'\tau}C(\mu)}e^{-(\mu'/E)(\sigma-2a)}}$$
$$=\int_0^\infty dx\int_x^\infty dx' e^{-\mu/Ex-\mu'/Ex'}\exp\left\{\frac{1}{\ell-\sigma}\right.$$
$$\left.\times[e^{-\beta(\mu+\mu')}x+e^{-\beta(\mu')}(x'-x)-x']\right\}. \tag{44}$$

Evaluating the integral, we finally arrive at

$$\Lambda(\mu,\mu')=\frac{\overline{C(\mu')\overline{e^{-\mu'\tau}C(\mu)}e^{-(\mu'/E)(\sigma-2a)}}}{\left[\frac{\mu'}{E}(\ell-\sigma)+1-e^{-\beta(\mu')}\right]\left[\frac{\mu+\mu'}{E}(\ell-\sigma)+1-e^{-\beta(\mu+\mu')}\right]}$$
$$=e^{-(\mu'/E)(\sigma-2a)}[\overline{e^{-\mu'\tau}C(\mu)}/\overline{C(\mu+\mu')}]B(\mu')B(\mu+\mu'). \tag{45}$$

For two random processes $f(t)$ and $g(t)$, we introduce the notation

$$K_{f|g}(\mu,\mu')=\int_0^\infty\int_0^\infty dt dt' e^{-\mu t-\mu' t'}[\overline{f(t)g(t')}-\overline{f(t)}\,\overline{g(t')}] \tag{46}$$

provided the double integral exists. From Eq. (24) and the subsequent computation, we find in the limit of N going to infinity

$$K_{\dot{x}|\dot{x}}(\mu,\mu')=\frac{B(\mu+\mu')}{\mu\mu'\overline{C(\mu+\mu')}}[\overline{C(\mu)C(\mu')}+e^{-(\mu'/E)(\sigma-2a)}$$
$$\times\overline{e^{-\mu'\tau}C(\mu)}B(\mu')+e^{-(\mu/E)(\sigma-2a)}$$
$$\times\overline{e^{-\mu\tau}C(\mu')}B(\mu)]-\frac{B(\mu)B(\mu')}{\mu\mu'}. \tag{47}$$

Set now $\mu=\epsilon/2+i\omega$ and $\mu'=\mu^*$. The velocity power spectrum is

$$S_{\dot{x}|\dot{x}}(\omega)=\lim_{\epsilon\to0}\epsilon K_{\dot{x}|\dot{x}}(\mu,\mu^*)$$
$$=\lim_{\epsilon\to0}\epsilon\left|\int_0^\infty e^{-\mu t}[\dot{x}(t)-\overline{\dot{x}(t)}]dt\right|^2$$
$$=\left(\lim_{\epsilon\to0}\frac{\epsilon B(\epsilon)}{\overline{C(\epsilon)}}\right)\frac{1}{\omega^2}\{\overline{|C(i\omega)|^2}$$
$$+2\mathrm{Re}[e^{(i\omega/E)(\sigma-2a)}B(-i\omega)\overline{e^{i\omega\tau}C(i\omega)}]\}$$
$$=\frac{E}{\ell-\overline{A}}\frac{1}{\omega^2}\left\{\overline{|C(i\omega)|^2}\right.$$
$$\left.+2\mathrm{Re}\left[\frac{\overline{C(-i\omega)e^{i(\omega/E)(\sigma-A)}C(i\omega)}}{1-\overline{e^{i(\omega/E)(\sigma-A)}}-i\frac{\omega}{E}(\ell-\sigma)}\right]\right\}, \tag{48}$$

where

$$A=2a-E\tau. \tag{49}$$

We recall that $C(\mu)$ was defined through Eqs. (19), (17), and (11), and in the last line of Eq. (48) the bar stands for averaging over the remaining (single-scatterer) randomness. Equation (48) is the main result of our paper. For ω real, $S_{\dot{x}|\dot{x}}(\omega)$ is a real, even and nonnegative function. If $E>E_c=\sup_{j,x}\phi_j'(x)$, the passage times τ_j are distributed on a bounded support, the velocity correlations decay exponentially and $S_{\dot{x}|\dot{x}}(\omega)$ is also analytic at $\omega=0$. In this case, the asymptotic displacement of the particle is a drift with a superimposed normal diffusion. Indeed, the diffusion constant D is given by

$$2D=\lim_{\epsilon\downarrow0}\epsilon^2\int_0^\infty e^{-\epsilon t}\overline{[x(t)-\overline{x(t)}]^2}dt$$
$$=\lim_{\epsilon\downarrow0}\int_0^\infty\frac{\epsilon K_{\dot{x}|\dot{x}}\left[\frac{\epsilon}{2}(1+iy),\frac{\epsilon}{2}(1-iy)\right]}{1+y^2}\frac{dy}{\pi}=S_{\dot{x}|\dot{x}}(0). \tag{50}$$

Notice that $\epsilon K_{\dot{x}|\dot{x}}$ in the integrand has a uniform upper bound. By using Eqs. (22), (23), (31), (33), and (36), a somewhat tedious computation yields

$$2D = S_{\dot{x}|\dot{x}}(0) = \frac{E}{(\ell - \overline{A})^3}\left[\ell^2(\overline{A^2 - \overline{A}^2}) + (\ell-\sigma)^2\overline{A}^2\right]. \tag{51}$$

Thus, D depends on the randomness through the averages ℓ, $\overline{\tau}$, and $\overline{\tau^2}$. We conclude that in our example of a deterministic dynamics, with a uniformly distributed set of quenched random scatterers, there is normal diffusion. If $E = \sup_{j,x}\phi_j'(x)$, we can loose normal diffusion. We will discuss this phenomenon later on.

B. A crystal of scatterers

Here, we choose $r_j = L_0 + (j-1)\sigma$, that is, the scatterers are placed equidistantly. Because now $r_{j+1} - r_j \equiv \sigma$, in the limit when N goes to infinity we also obtain $\ell = \sigma$. This can directly be substituted in Eqs. (37), (48), and (51) to obtain

$$\overline{\dot{x}(\infty)} = \frac{E\sigma}{\sigma + E\overline{\tau} - 2a}, \tag{52}$$

$$S_{\dot{x}|\dot{x}}(\omega) = \frac{E}{\sigma - \overline{A}}\frac{1}{\omega^2}\left\{\overline{|C(i\omega)|^2} + 2\mathrm{Re}\left[\frac{\overline{C(-i\omega)\,e^{i(\omega/E)(\sigma-A)}C(i\omega)}}{1 - \overline{e^{i(\omega/E)(\sigma-A)}}}\right]\right\}, \tag{53}$$

and

$$S_{\dot{x}|\dot{x}}(0) = E\sigma^2\frac{\overline{A^2} - \overline{A}^2}{(\sigma - \overline{A})^3}. \tag{54}$$

We note that substitution of $\ell = \sigma$ in Eqs. (47) and (48) eliminates, at least asymptotically, the randomness of r_j [still admitting $x_N = o(L)$]. If we also drop averaging over the potentials ϕ_j, all randomness is lifted, and by the general definition (46), $K_{\dot{x}|\dot{x}}$ and $S_{\dot{x}|\dot{x}}$ must identically vanish. It can be easily verified that formulas (47) and (48) or (53), and also (51), indeed show this property.

IV. FROM CONDUCTION TO LOCALIZATION

When E varies continuously and passes the value $E_c = \sup_{j,x}\phi_j'(x)$, it switches between a conducting state for $E > E_c$ and an isolating, or localizing, state for $E < E_c$. At $E = E_c$ there may be localization, if $\overline{\tau} = \infty$ [see Eq. (37)], and there may also be conduction if $\overline{\tau}$ remains finite. In both cases there may still occur a large variety of different situations, characterized by different critical exponents and normal or anomalous diffusion. The transition between conduction and localization bears a resemblence with phase transitions. For instance, E can be considered as the analog of the temperature T, the region $E > E_c$ that of $T < T_c$, and $\overline{\dot{x}(\infty)}$ may correspond, e.g., to the spontaneous magnetization and the diffusion constant to the static zero-field magnetic susceptibility. We may have first- and second-order transitions and varying exponents depending on the form of probability distribution of the passage time τ.

Let us discuss a simple example which illustrates the different possibilities. We consider

$$\phi_j(x) = f_j \times (|x| - a), \quad |x| \leqslant a, \tag{55}$$

so that $\phi_j'(x) = \pm f_j$. Then

$$\tau_j = a\left(\frac{1}{E - f_j} + \frac{1}{E + f_j}\right). \tag{56}$$

Suppose that the common probability density of the random forces f_j has a bounded support $[b,c]$, where $c > 0$ and $|b| < c$. Boundedness is needed to have a transition, and with the above choice $E_c = c$. Let us consider on this support a one-parameter family of probability densities

$$p_\gamma(u) = \frac{(\gamma+1)(c-u)^\gamma}{(c-b)^{\gamma+1}}, \tag{57}$$

with $\gamma > -1$. For $E > c$,

$$\overline{\tau^n} = \frac{(\gamma+1)a^n}{(c-b)^{\gamma+1}}\int_0^{c-b} v^\gamma\left[\frac{1}{E+c-v} + \frac{1}{v+\varepsilon}\right]^n dv, \tag{58}$$

where we have introduced the short-hand notation $\varepsilon = E - c$. As ε goes to zero, asymptotically

$$\overline{\tau^n} = \begin{cases} \dfrac{(\gamma+1)a^n}{(c-b)^{\gamma+1}}\displaystyle\int_0^{c-b} v^\gamma\left[\frac{1}{2c-v} + \frac{1}{v}\right]^n dv + o(1), & n < \gamma+1 \\[2ex] \dfrac{(\gamma+1)a^{\gamma+1}}{(c-b)^{\gamma+1}}\ln\dfrac{c-b}{\varepsilon} + O(1), & n = \gamma+1 \\[2ex] \dfrac{(\gamma+1)a^n}{(c-b)^{\gamma+1}}\dfrac{\varepsilon^{-n+\gamma+1}}{n-\gamma-1} + O(\varepsilon^{-n+\gamma+2}), & n > \gamma+1. \end{cases} \tag{59}$$

In particular,

$$\overline{\tau}=\begin{cases}\dfrac{(\gamma+1)a}{(c-b)^{\gamma+1}}\displaystyle\int_0^{c-b}v^{\gamma}\left[\frac{1}{2c-v}+\frac{1}{v}\right]dv+o(1), & \gamma>0\\ \dfrac{a}{c-b}\ln\dfrac{c-b}{\varepsilon}+O(1), & \gamma=0\\ \dfrac{(1-|\gamma|)a}{|\gamma|(c-b)^{1-|\gamma|}}\varepsilon^{-|\gamma|}+O(\varepsilon^{1-|\gamma|}), & \gamma<0,\end{cases} \tag{60}$$

and

$$\overline{\tau^2}=\begin{cases}\dfrac{(\gamma+1)a^2}{(c-b)^{\gamma+1}}\displaystyle\int_0^{c-b}v^{\gamma}\left[\frac{1}{2c-v}+\frac{1}{v}\right]^2dv+o(1), & \gamma>1\\ \dfrac{(2a^2}{(c-b)^2}\ln\dfrac{c-b}{\varepsilon}+O(1), & \gamma=1\\ \dfrac{(\gamma+1)a^2}{(c-b)^{\gamma+1}}\dfrac{\varepsilon^{-1+\gamma}}{1-\gamma}+O(\varepsilon^{\gamma}), & \gamma<1.\end{cases} \tag{61}$$

If we are interested in quantities depending on the random potentials only via ℓ, $\overline{\tau}$, and $\overline{\tau^2}$, as $\overline{\dot{x}(\infty)}$ and $S_{\dot{x}|\dot{x}}(0)$, we can distinguish the following cases.

(1) $\gamma>1$. In this case $\overline{\tau}$ and $\overline{\tau^2}$ have a finite limit as $E\downarrow E_c$. As a consequence, $\overline{\dot{x}(\infty)}$ is positive and $S_{\dot{x}|\dot{x}}(0)$ is finite at $E=E_c$ and their value is given, respectively, by Eqs. (37) and (51): there is conduction with normal diffusion. So when E increases and goes through E_c, both $\overline{\dot{x}(\infty)}$ and $S_{\dot{x}|\dot{x}}(0)$ change discontinuously from 0 to a positive value and then vary continuously. This is analogous with a first-order phase transition.

(2) $0<\gamma\leqslant 1$. From the point of view of $\overline{\dot{x}(\infty)}$ the transition is still of first order, but the divergence of D,

$$D\approx\frac{E^3\ell^2\overline{\tau^2}}{2(\ell-\overline{A})^3}, \tag{62}$$

with the diverging $\overline{\tau^2}$ when $E\downarrow E_c$ resembles the divergence of the susceptibility at T_c in second-order magnetic phase transitions. Thus, in this case we have conduction accompanied with an anomalous diffusion.

(3) $-1<\gamma\leqslant 0$. Both $\overline{\tau}$ and $\overline{\tau^2}$ diverge when $E\downarrow E_c$, so $\overline{\dot{x}(\infty)}$ tends to zero and

$$D\approx\frac{\ell^2\overline{\tau^2}}{2\overline{\tau^3}}\propto\varepsilon^{-1-2\gamma}, \tag{63}$$

with an additional factor $|\ln\varepsilon|^{-3}$ if $\gamma=0$. So D diverges if $\gamma>-1/2$, tends to zero if $-1<\gamma<-1/2$ and to a finite nonzero limit if $\gamma=-1/2$.

Let us emphasize that the probability density p_γ is purely continuous and, thus, the probability that $f_j=E_c$ for a given j is zero. The probability that $f_j=E_c$ for *any* j is still zero. So with probability 1 the particle will never be stopped, and $\overline{\dot{x}(\infty)}=0$ means only that, with probability 1, $x(t)=o(t)$, i.e., $x(t)$ increases slower than t. It is in this way that we can understand the different possibilities of diffusion in the third case.

V. SCALING AT CRITICALITY

When $E>E_c$, $S_{\dot{x}|\dot{x}}(\omega)$ is a meromorphic function of ω with no pole in a neighborhood of the origin. Analyticity at $\omega=0$ will be lost as E attains its critical value E_c. In this section, we derive a scaling law describing the behavior of $S_{\dot{x}|\dot{x}}(\omega;E)$ when $\omega\to 0$ and $E\downarrow E_c$ simultaneously in such a way that $z=\varepsilon/\omega\equiv(E-E_c)/\omega$ is kept fixed. More specifically, we expect that in this limit

$$S_{\dot{x}|\dot{x}}(\omega;E)\approx\frac{r(z)}{\omega}. \tag{64}$$

Below we prove the above form and find the scaling function $r(z)$. The model that we use for explicit computations is the same as in Sec. IV, given by Eqs. (55) and (57), although the conclusions certainly hold more generally. First, we would like to give an argument why one can expect the asymptotic form (64). If $\varepsilon>0$, the correlation function

$$\xi(t,t')=\overline{\dot{x}(t)\dot{x}(t')-\overline{\dot{x}(t)}\dot{x}(t')} \tag{65}$$

decays exponentially with $|t-t'|$, and the correlation time can be approximated with $\tau_m=2aE/\varepsilon(E+E_c)$, the maximum passage time through a scatterer [see Eq. (56)]. This allows one to write down at least two different, qualitatively reasonable, approximations to $\xi(t,t')$, namely,

$$\xi_1(t,t')=\xi_0\Theta(\tau_m-|t-t'|) \quad \text{and} \quad \xi_2(t,t')=\xi_0 e^{-|t-t'|/\tau_m}, \tag{66}$$

with $\Theta(y)$ being the Heaviside function. Both lead to a form like Eq. (64). From ξ_1, we obtain

$$S^{(1)}_{x|x}(\omega;E)\approx\frac{2\xi_0}{\omega}\sin\omega\tau_m\approx\frac{2\xi_0}{\omega}\sin\frac{a}{z}, \tag{67}$$

while ξ_2 yields

$$S^{(2)}_{x|x}(\omega;E)\approx\frac{2\xi_0\tau_m^{-1}}{\tau_m^{-2}+\omega^2}\approx\frac{2\xi_0}{\omega}\frac{a/z}{1+(a/z)^2}. \tag{68}$$

In deriving $r(z)$, we will consider only negative values of γ, so that $0<\alpha\equiv-\gamma<1$, choose $b=0$ for the sake of simplicity and use the notation

$$P(y)=p_\gamma(E_c-y)=py^{-\alpha}, \tag{69}$$

cf. Eq. (57). We note that only

$$\sup_y y^\alpha P(y)<\infty \quad \text{and} \quad \lim_{y\to 0} y^\alpha P(y)=p, \tag{70}$$

where p is defined by the normalization, will be used below, so instead of Eq. (69) we can take any $P(y)$ with the properties (70). Notice that $\int_0^{E_c}dy(P(y)/y)=\infty$. Accordingly, we obtain

$$\overline{e^{i\omega\tau}}=\int_0^{E_c}dyP(y)\exp\left[i\omega a\left(\frac{1}{\varepsilon+y}+\frac{1}{\varepsilon+2E_c-y}\right)\right] \tag{71}$$

and

$$\lim_{\omega\to 0}\frac{(\overline{e^{i\omega\tau}}-1)}{\omega^{1-\alpha}}=p\int_0^\infty ds s^{-\alpha}\left[\exp\left(i\frac{a}{s+z}\right)-1\right]:=g(z). \tag{72}$$

Furthermore, from the third line of Eq. (60), we have

$$\overline{\tau}=\tau_0\varepsilon^{-\alpha}+O(\varepsilon^{1-\alpha}) \tag{73}$$

as $\varepsilon\to 0$. In order to compute the current spectrum other terms have to be computed, like

$$1-e^{-\beta(i\omega)}=1-\overline{e^{-i(\omega/E)(\sigma-A)}}\approx-\omega^{1-\alpha}\overline{g(z).} \tag{74}$$

Next, we need the average of

$$C(\mu)=\mu\int_{-a}^{a}dye^{-\mu\vartheta(y)}-E(1-e^{-\mu\tau}). \tag{75}$$

For the potential (55)

$$\vartheta(y)=\int_{-a}^{y}\frac{ds}{E-f}=\Theta(-y)\frac{(a+y)}{E+f}+\Theta(y)\frac{a}{E+f}+\frac{y}{E-f}. \tag{76}$$

Inserting Eq. (76) in Eq. (75),

$$C(\mu)=f\left[1-2\exp\left(-\frac{\mu a}{E+f}\right)+\exp\left\{-\mu a\left(\frac{1}{E+f}+\frac{1}{E-f}\right)\right\}\right], \tag{77}$$

so that

$$\overline{C(\mu)}=\int_0^{E_c}dyP(y)(E_c-y)\left[1-2\exp\left(-\frac{\mu a}{\varepsilon+2E_c-y}\right)+\exp\left(-\frac{\mu a}{\varepsilon+y}-\frac{\mu a}{\varepsilon+2E_c-y}\right)\right] \tag{78}$$

and

$$\overline{e^{\mu\tau}C(\mu)}=\int_0^{E_c}dyP(y)(E_c-y)\left[\exp\left(\frac{\mu a}{\varepsilon+y}+\frac{\mu a}{\varepsilon+2E_c-y}\right)-2\exp\left(\frac{\mu a}{\varepsilon-y}\right)+1\right]. \tag{79}$$

Accordingly, in the limit of vanishing ω

$$\omega^{\alpha-1}\overline{C(i\omega)\to p}E_c\int_0^\infty dss^{-\alpha}\left[\exp\left(-i\frac{a}{s+z}\right)-1\right]=E_c g^*(z) \tag{80}$$

and

$$\omega^{\alpha-1}\overline{e^{i\omega\tau}C(i\omega)}\to pE_c\int_0^\infty dss^{-\alpha}\left[\exp\left(i\frac{a}{s+z}\right)-1\right]=E_c g(z). \tag{81}$$

Moreover, one has (taking $\ell=\sigma$ for simplicity)

$$B(-i\omega)=\frac{\overline{C(-i\omega)}}{1-\exp[-\beta(i\omega)]}=\left[-\frac{\omega^{\alpha-1}\overline{C(i\omega)}}{\omega^{\alpha-1}\{1-\exp[-\beta(i\omega)]\}}\right]^*\to\left[-\frac{E_c g^*(z)}{g^*(z)}\right]^*=-E_c. \tag{82}$$

Similar expressions have to be obtained for

$$\overline{|C(i\omega)|^2}=\int_0^{E_c}dyP(y)(E_c-y)^2\left|1-2\exp\left(-i\frac{\omega a}{\varepsilon+2E_c-y}\right)+\exp\left(-i\frac{\omega a}{\varepsilon+y}-i\frac{\omega a}{\varepsilon+2E_c-y}\right)\right|^2, \tag{83}$$

so that

$$\omega^{\alpha-1}\overline{|C(i\omega)|^2}\to pE_c^2\int_0^\infty dss^{-\alpha}\left|\exp\left(-i\frac{a}{s+z}\right)-1\right|^2$$
$$=E_c^2u(z). \tag{84}$$

By substituting into Eq. (48), we finally obtain

$$S_{\dot{x}|\dot{x}}(\omega)=\frac{\varepsilon+E_c}{\omega^2[\ell-2a+(\varepsilon+E_c)\overline{\tau}]}\{\overline{|C(i\omega)|^2}$$
$$+2\mathrm{Re}[e^{i\omega\delta}B(-i\omega)\overline{e^{i\omega\tau}C(i\omega)}]\}$$
$$\approx\frac{\varepsilon+E_c}{\omega^2\omega^{\alpha-1}[\ell-2a+(\varepsilon+E_c)\tau_0\varepsilon^{-\alpha}]}E_c^2[u(z)$$
$$-2\mathrm{Re}g(z)]. \tag{85}$$

Accordingly, in the limit of small ε one obtains

$$\omega^{\alpha+1}\varepsilon^{-\alpha}S_{\dot{x}|\dot{x}}(\omega)\to\frac{E_c^2}{\tau_0}[u(z)-2\mathrm{Re}g(z)]$$
$$=\frac{E_c^2}{\tau_0}p\int_0^\infty dtt^{-\alpha}[|1-e^{-i(a/t+z)}|^2$$
$$-2\ \mathrm{Re}(e^{i(a/t+z)}-1)]$$
$$=\frac{4E_c^2}{\tau_0}p\int_0^\infty dtt^{-\alpha}\left[1-\cos\left(\frac{a}{t+z}\right)\right]. \tag{86}$$

Finally,

$$\lim_{\omega\to0}\omega S_{\dot{x}|\dot{x}}(\omega;E_c+\omega z)=\frac{4E_c^2z^\alpha\int_0^\infty dtt^{-\alpha}\left[1-\cos\left(\frac{a}{t+z}\right)\right]}{a\int_0^\infty dtt^{-\alpha}(1+t)^{-1}}$$
$$\equiv r(z). \tag{87}$$

It is worth pointing out the relevance of this result: a ω^{-1} component emerges naturally at criticality in the spectral properties of a purely mechanical model with disorder.

VI. CONCLUSION

In this paper, we have studied the motion of an overdamped particle along a line under the influence of a constant external field and a random sequence of scatterers.

This simple mechanical model has several virtues. The first one is mainly pedagogical, since it can be solved exactly: analytic formulas have been obtained for the time average of the velocity and for the Fourier transform of its time-autocorrelation function—the velocity power spectrum. We have found that above a critical value of the external field the asymptotic motion of the particle is a drift with a superimposed normal diffusion. The diffusion constant can be also calculated explicitly. Moreover, at the critical field, we have found different kinds of anomalous diffusion, depending on the probability density associated with the random potential of the scatterers.

The conclusion that normal and anomalous diffusive behavior can be obtained from a purely mechanical description, by means of the randomness of the potential, and not by introducing any stochastic or periodic time-dependent forcing is the main conceptual virtue of this model.

In a physical perspective, there are different ways in which it can be interpreted. For instance, in order to avoid a too formal mathematical treatment from the very beginning we have presented the model as describing the motion of an overdamped charged particle subject to a constant electric field. Since below the critical value of the electric field the current vanishes, the model can be interpreted as a classical representation of the transition from an insulator to a conductor state, controlled by the electric field. By analogy with the theory of phase transitions, we have shown that this transition can be first or second order and the corresponding critical properties are determined by the scaling law characterizing the power spectrum of the current. As discussed in the Appendix, another interpretation of our model corresponds to the motion of a massive particle submitted to strong friction, sitting or sliding on a rough surface, inclined with respect to the ground. In the two-dimensional version of this model the roughness of the surface can be modeled by a lattice of random harmonic traps and the component of the gravity force parallel to the surface plays the role of the driving field. We have shown that this dynamics is asymptotically described by the one-dimensional model introduced in this paper. Accordingly, all the above described scenario is recovered. In particular, anomalous diffusion properties are found to characterize the dynamics at the critical value of the inclination angle. This is quite important for the understanding of the the phenomenon of segregation by flow, that has been recently analyzed by numerical [11] and experimental studies [12].

As a final remark, it is worth mentioning that a real limitation of our studies is that we have completely neglected inertial effects. For what concerns the one-dimensional models studied in this paper it appears possible to extend our studies to the case of a dynamics described by Newton equations. Nonetheless, this seems a much more challenging goal for the two-dimensional problem of the particle sliding on the rough, inclined surface.

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APPENDIX: A PARTICLE ON AN INCLINED PLANE

Let us consider a square lattice of disks of radius *a* and lattice constant *d*. A particle submitted to a very strong friction force moves in this lattice. Outside the disks it feels a constant positive force **F** in the x direction. Once it enters a disk, it is submitted, beside **F**, to a harmonic potential concentric with the disk and of a frequency ω depending on the disk. The frequencies are identically distributed independent random variables.

The dynamics is described by the equations

$$\gamma m\dot{\mathbf{r}}=\mathbf{F}-\sum_{\mathbf{n}\in Z^2}\nabla V_{\mathbf{n}}(r-d\mathbf{n}), \tag{A1}$$

$$V_{\mathbf{n}}(\mathbf{r})=\frac{1}{2}m\omega_n^2[(x-dn_x)^2+(y-dn_y)^2-a^2] \times\Theta(a-|\mathbf{r}-d\mathbf{n}|), \tag{A2}$$

where Θ is the Heaviside function. In an electric interpretation $\mathbf{F}=(qE,0)$, E being the electric field and q the charge of the particle. If we interprete the constant force as the result of the gravitational force on the particle, moving in the plane inclined with an angle α with respect to the horizontal then $\mathbf{F}=(mg\sin\alpha,0)$.

We can suppose that initially the particle is located at some point between the disks. Two possibilities arise. The trivial first one is that the particle does not hit the disks: its motion is a straight line in the x direction. The second possibility is that it hits first a disk at its right. Then, we can easily see that it moves only along the row of disks at the right of the first one and, therefore, its motion is restricted to this row. We select the row passing through the origin. Let a and γ^{-1} be the unit of length and time, respectively. The nth disk of the row is centered at $[r_n=(d+2)n,0]$. Inside it the dynamics is given by

$$\dot{x}=-\left(\frac{\omega_n}{\gamma}\right)^2(x-r_n)+\frac{F}{\gamma^2 ma}, \tag{A3}$$

$$\dot{y}=-\left(\frac{\omega_n}{\gamma}\right)^2 y. \tag{A4}$$

If

$$\epsilon_n=\frac{F}{ma\omega_n^2}<1, \tag{A5}$$

once the particle enters this disk, it stays in it forever and reaches the equilibrium point $(r_n+\epsilon_n,0)$ in infinite time. If, on the other hand, $\epsilon_n>1$, it goes across the disk and, if it entered the disk in time t_n at the point (x_n,y_n), it leaves it in time t_n' at (x_n',y_n'), where

$$x_n'-r_n-\epsilon_n=(x_n-r_n-\epsilon)e^{-(\omega_n/\gamma)^2(t_n'-t_n)}, \tag{A6}$$

$$y_n'=y_n e^{-(\omega_n/\gamma)^2(t_n'-t_n)}, \tag{A7}$$

so that the trajectory is a straight line

$$\frac{x_n'-r_n-\epsilon_n}{x_n-r_n-\epsilon_n}=\frac{y_n'}{y_n}. \tag{A8}$$

If $y_n=0$ then $y_n'=0$ and we use Eq. (A6) to determine x_n'. It is worth recalling that the following relation holds

$$(x_n-r_n)^2+y_n^2=(x_n'-r_n)^2+(y_n')^2=1.$$

After time t_n', the particle reaches (x_{n+1},y_{n+1}), where

$$y_{n+1}=y_n'. \tag{A9}$$

Note that $|y_n'|<|y_n|$, so that the particle will necessarily go along the row. Equations (A8) and (A9) give

$$y_{n+1}=\frac{y_n(\epsilon_n^2-1)}{\epsilon_n^2+1-2\epsilon_n(x_n-r_n)}=\frac{y_n(\epsilon_n^2-1)}{\epsilon_n^2+1+2\epsilon_n\sqrt{1-y_n^2}}, \tag{A10}$$

because x_n-r_n must be negative if the crossing takes place.

Let $\Omega=\max\omega_n<\infty$ and $F_c=ma\Omega^2$. If $F<F_c$,

$$\mathrm{Prob}\{\epsilon_n>1\}=\int_0^{\sqrt{F/ma}}\rho(\omega)d\omega=p<1$$

and

$$\mathrm{Prob}\{\text{at least one }\ \epsilon_n<1\}=1.$$

Therefore, if $F<F_c$, with probability 1 the particle is trapped. If $F\geqslant F_c$, then

$$\mathrm{Prob}\{\epsilon_n\geqslant 1\}=1, \qquad y_{n+1}\leqslant y_n\frac{\epsilon_n^2-1}{\epsilon_n^2+1}. \tag{A11}$$

If $\omega_n>\omega>0$, then

$$\epsilon_n<\frac{F}{ma\omega^2}=\hat{\epsilon}.$$

Thus,

$$\frac{\epsilon_n^2-1}{\epsilon_n^2+1}\leqslant\frac{\hat{\epsilon}^2-1}{\hat{\epsilon}^2+1}=\delta<1 \tag{A12}$$

and $y_N\leqslant y_0\delta^N\to 0$. Thus, asymptotically the dynamics becomes purely one-dimensional, i.e., restricted to x and, therefore, to our mechanical model of diffusion. For a harmonic trap of frequency ω (and setting $m=1$) the passage time τ is given by

$$\tau=\frac{1}{\omega^2}\ln\frac{E+\omega^2 a}{E-\omega^2 a}.$$

Now because $E_c=\Omega^2 a$, $\overline{\tau}$ and $\overline{\tau^2}$ remain finite at $E=E_c$ and, hence, the transition is of first order.

An interesting extension of the model is obtained by admitting the force **F** to be in an arbitrary direction **e** $=(\cos\vartheta,\sin\vartheta)$. Decomposing **r** in parallel and perpendicular components, the perpendicular component is always attenuated by the friction force while the parallel one may increase globally linearly if $F>F_c$. If ϑ/π is irrational, the positions of the traps met by the particle form a quasiperiodic sequence. Even without the randomness of the frequency this may have interesting effects on the diffusion.

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